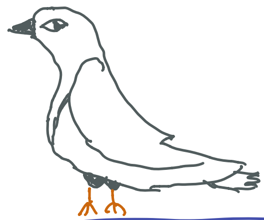


Pigeonhole Principle and Some Theorems

Ch. 4 of Extremal Combinatorics – Jukna

Rohan Garg

The pigeonhole principle: "$n+1$ pigeons, n holes. The pigeons can't sit in the n holes such that every pigeon is alone."

More generally: If a set of more than kn objects is partitioned into n classes, then some class must have more than k objects.

Warm-up:

Def. degree of vertex = # edges adjacent to it.

Claim: In any graph, there exist two vertices with the same degree.

Proof: Given undirected graph G on n vertices.

make n pigeonholes $\longleftrightarrow$ degree. $0, 1, \ldots, n-1$

We put vertex v into hole k if $\deg(v) = k$.

If we wanted all n vertices to have different degree, then every hole has one vertex.

one vertex^x in "0" hole.
one vertex^y in "n-1" hole. } Contradiction. $\square$

<u>Def.</u> $\alpha(G)$ the independence number of graph G. The max number of pairwise non-adjacent vertices. (max ind. set)

<u>Def</u> $\chi(G)$ the chromatic number of G. $\chi(G)$ is the minimum number of colors in a vertex-coloring of G such that no two adjacent vertices have the same color.

<u>Claim:</u> In any graph G with n vertices, $n \leq \alpha(G) \cdot \chi(G)$

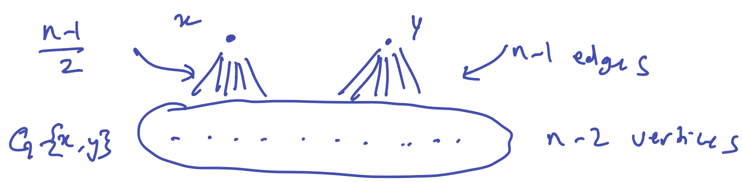
<u>Proof:</u>
- Partition the vertices into $\chi(G)$ color classes.
- By pigeonhole principle, some color class will have more than $n/\chi(G)$ vertices.
- So, these vertices, by def. of $\chi(G)$, are pairwise non-adjacent.

$$\Rightarrow \alpha(G) \geq n/\chi(G) \longrightarrow n \leq \alpha(G) \cdot \chi(G).$$

<u>Claim:</u> Let G be an n-vertex graph. If every vertex

has degree at least $\frac{(n-1)}{2}$, then the graph is connected.

Proof: Take any two vertices x & y. If these vertices don't have an edge between them, we know there are $n-1$ edges to the rest of the graph with $n-2$ vertices.



So, by PHP, there is a shared vertex in those $n-2$ vertices.

So, there is a path from $x \to y$.

$\square$

The Erdös-Szekeres Theorem:

Let $A = (a_1, a_2, \ldots, a_n)$ be a sequence of different numbers. B is a subsequence of A of k terms if

$B = (a_{i_1}, a_{i_2}, \ldots, a_{i_k})$ where the elements of B appear in the same order as they do in A.

It feels like there's a tradeoff between length of longest increasing subsequence and longest decreasing subsequence.

Theorem (1935): $A = (a_1 \ldots a_n)$ be a sequence of different reals.

If $n \geq sr+1$, then either A has an increasing subsequence of $s+1$ terms or A has a decreasing subsequence of $r+1$ terms (or both).

<u>Proof (Seidenberg 1959)</u>: Assign to each a_i a score $= (x_i, y_i)$.

x_i = # terms in longest increasing subsequence <u>ending at a_i</u>

y_i = # terms in longest decreasing subsequence <u>starting at a_i</u>.

Show, $i \neq j \rightarrow (x_i, y_i) \neq (x_j, y_j)$.

$\ldots\ldots a_i \ldots a_j \ldots\ldots$:

$a_i < a_j \rightarrow x_j > x_i$. Longest inc. subsequence ending at a_j > that of a_i.

$a_i > a_j \rightarrow y_i > y_j$.

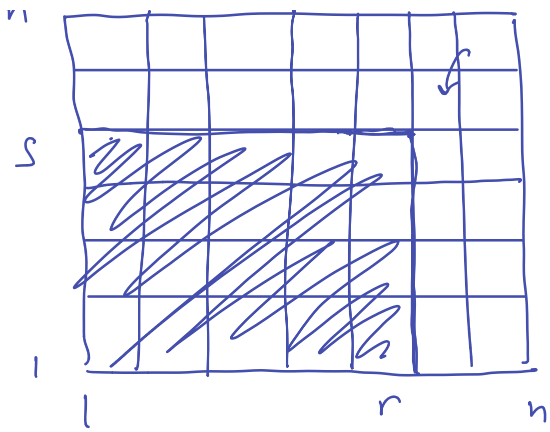
I can make a longer decreasing subsequence for a_i than a_j by adding a_i to the start of a_j's sequence.

$A = (1, 3, 5, 2, 4)$

3 2

$|A| = n \geq sr+1$.

Grid of n^2 pigeon holes:



Put a_i into hole (x_i, y_i) in the grid.

$1 \le x_i, y_i \le n \ . \ \forall i.$ $\quad i \neq j \longrightarrow (x_i, y_i) \neq (x_j, y_j)$.

either its $x_i > r$ or $y_i > s$.

$\Longrightarrow$. Longest increasing subsequence is $\geq r+1$

or

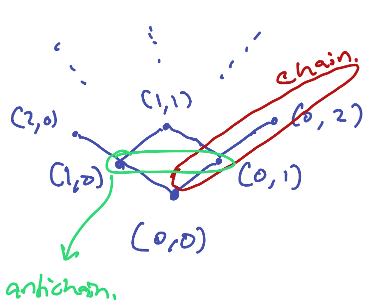
longest decreasing subsequence is $\geq s+1$.

A (weak) partial order on a set P is a binary relation $<$ on its elements. x & y are comparable if $x < y$ or $y < x$ (or both.).

Partially ordered set = Poset.

- A chain $Y \subseteq P$ is a set where all elements in Y are comparable.

- An antichain $Y \subseteq P$ is a set where all elements in Y are incomparable.



Lemma (Dilworth 1950): In any partial order on a set P of $n \geq sr+1$ elements, there exists either a chain of length $s+1$ or an antichain of length $r+1$.

Proof: Suppose there is no chain of length $s+1$. We'll define a function $f: P \rightarrow \{1 \ldots\ldots s\}$.
↳ s classes

$f(x)$ = the maximal # of elements in a chain where x is the greatest element.

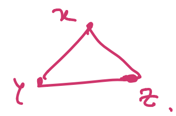
By the pigeonhole principle, some class will have at least $r+1$ elements. By the def. of f, these elements are incomparable. So, there is some antichain of length at least $r+1$.

□

Triangles in graphs:

"How many edges are possible in a triangle free graph on $2n$ vertices?"

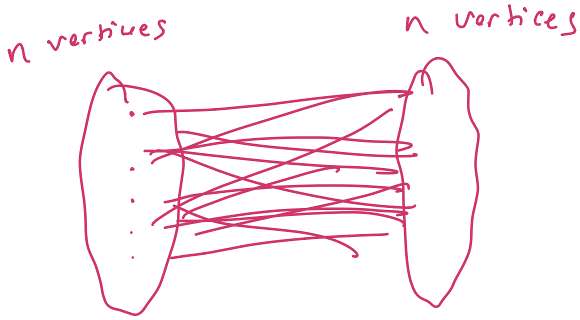
Triangle = $\{x, y, z\}$ with edges between x, y & z.



2n vertices → 2n−1 edges spanning tree → no triangles.

2n edges in a big cycle.

2n vertices as a bipartite graph.



→ n^2 edges.

max possible # edges in a triangle free graph.

Theorem: (Mantel 1907): If a graph on 2n vertices has n^2+1 edges, then G contains a triangle.

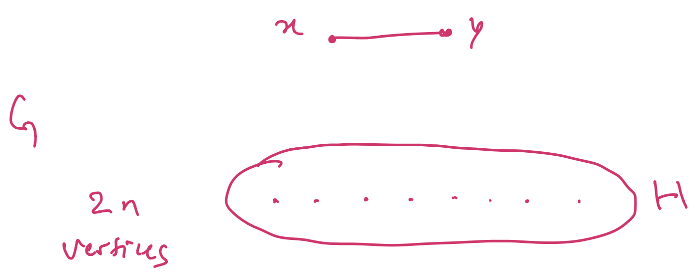
Proof: Inductively.

Base case: $n=1$. → 2 vertices. Can't have 2 edges. Statement is true.

Now, we'll assume it is true for n and we'll consider a graph on $2(n+1)$ vertices with $\underline{(n+1)^2+1}$ edges.
$2n+2$

Let x, y be adjacent vertices in G.

Let H be the remaining subgraph.



If H has $\geq n^2+1$ edges $\to$ H has a triangle
we're done.

Suppose H has at most n^2 edges. Then.

$$(n+1)^2+1 = n^2+2n+2 \quad \text{total edges.}$$

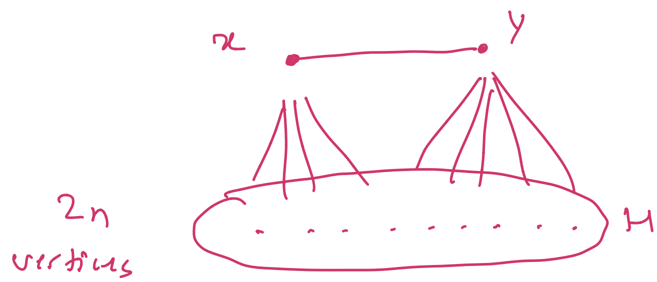
$$- \quad n^2 \quad \text{edges in } H$$

$$= 2n+2 \quad \text{edges}$$

$$- \quad 1 \quad \text{edge } (x,y)$$

$$= 2n+1 \quad \text{edges between } x \,\&\, y$$

and H



$2n+1$ edges to $2n$ vertices. So, there must be a shared vertex z. s.t the triangle $\{x, y, z\}$ is formed.

$\square$

$\Rightarrow$ lots of proofs of this!!!

Triangles = 3-clique

How does this generalize to k-cliques where $k \geq 3$??

<u>Turán's Theorem</u>: (1941) If a graph $G = (V, E)$ on n vertices has no $(k+1)$-clique for $k \geq 2$, then

$$|E| \leq \left(1 - \frac{1}{k}\right)\frac{n^2}{2} \qquad (T).$$

$k=2 \longrightarrow$ Mantel's Theorem.

Proof: Inductively on n.

$n=1$, trivially true. Base case. (T) is true for $n=1$.

$|E|=0 \le (1-\frac{1}{k})\frac{1}{2}$.

" $k=2 \to$ Mantel's Thm.

Suppose (T) is true for all graphs on at most $(n-1)$ vertices. Let $G=(V,E)$ be a graph on n vertices without $(k+1)$ cliques with a maximal number of edges.

$\to$ G must have some k-clique.

Let A be a k-clique and set $B=V-A$.

$e_A = \#$ edges inside of A.

$$= \binom{k}{2} = \frac{k \cdot k-1}{2}$$

$e_B = \#$ edges inside of B.

$e_{A,B} = \#$ edges across A and B.

$$e_B \le \left(1-\frac{1}{k}\right)\frac{(n-k)^2}{2} \leftarrow \text{from IH}$$

Since G has no $(k+1)$-clique, every $x \in B$

is connected to at most $k-1$ vertices in A.

$$e_{A,B} = \underbrace{(n-k)}_{\text{vertices in } B} \cdot (k-1).$$

<u>Identity:</u>

$$\left(1-\frac{1}{k}\right)\frac{n^2}{2} = \binom{k}{2}\left(\frac{n}{k}\right)^2$$

$$\frac{k \cdot k-1}{2} \cdot \frac{n^2}{k^2} = \frac{k^2-k}{2} \cdot \frac{n^2}{k^2}$$

$$= \frac{n^2}{2}\left(\frac{k^2-k}{k^2}\right) = \frac{n^2}{2}\left(1-\frac{1}{k}\right).$$

$$|E| \leq e_A + e_B + e_{A,B}$$

$$= \binom{k}{2} + \left(1-\frac{1}{k}\right)\frac{(n-k)^2}{2} + (n-k)\cdot\overbrace{\left(k-1 \cdot \frac{k}{2}\right)}^{\binom{k}{2}} \cdot \frac{2}{k}$$

$$= \binom{k}{2} + \binom{k}{2}\left(\frac{n-k}{k}\right)^2 + \binom{k}{2}(n-k)\left(\frac{2}{k}\right)$$

$$= \binom{k}{2}\left(1 + \left(\frac{n-k}{k}\right)^2 + \frac{2(n-k)}{k}\right)$$

$$= \binom{k}{2}\left(1 + \frac{n-k}{k}\right)^2$$

— identity

$$= \left(1 - \frac{1}{k}\right)\frac{n^2}{2}$$

$\square$

Extremal Graph Theory

– Extremal Combinatorics → Stasys Jukna.

Thanks !! :)