PURDUE

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Ch. 6 of Vazirani's Approx Alg.

Problem: Feedback Vertex Set

undirected graph $G = (V, E)$

$w : V \to \mathbb{R}_{\geq 0}$

Goal: Find a min-weight subset of V whose removal leaves G acyclic.

← order the edges of G in some arbitrary order.

Defs:

The characteristic vector of a simple cycle C is a vector in $GF[2]^m$, $m = |E|$. It has 1's in components corresponding to edges of

C, and 0's elsewhere.

$[0\ 1\ 1\ 1\ 1\ 0]$

The cycle space of G is the subspace of $GF[2]^m$ that is spanned by the char. vectors of all simple cycles in G.

The cyclomatic number of G, $cyc(G)$, is the dimension of this space.

$comps(G)$ is # connected components in G.

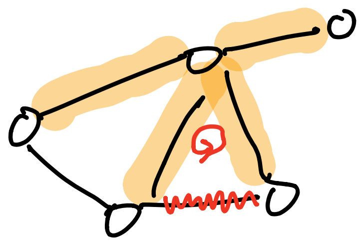
Thm: 6.2: $cyc(G) = |E| - |V| + comps(G)$

Pf:

1) Cycle space (of G) is sum of its connected comp. and so is the cyclomatic number.
So, we only consider a connected G.

Goal: $cyc(G) = |E| - |V| + 1$

→ Let T be a spanning tree in G. For each non tree edge e, define e's fundamental cycle to be $T \cup \{e\}$.



~~The set of~~ These char. vectors of all such fundamental cycles are linearly independent.

So: $cyc(G) \geq |E| - |V| + 1$

$$= |E| - (|V| - 1)$$

Each edge e in T defines a "fundamental cut" $(S, \bar{S})$.

Define characteristic vector of a cut to be a vector in $GF[2]^m$ where components corresponding to edges in the cut get 1's, 0's elsewhere.

Consider the $|V|-1$ vectors defined by edges of T.

Each cycle must cross each cut an even number of times. So, these vectors are orthogonal to the cycle space of G.

$$\begin{array}{cc} \text{CV of cut} & \text{CV of cycle} \\ \begin{bmatrix} \quad & \quad & \quad \end{bmatrix} & \begin{bmatrix} \\ \\ \\ \end{bmatrix} \end{array} = \underbrace{1+1+1+\dots}_{\text{even \# times}}\dots 1$$

GF:

+	0	1
0	0	1
1	1	0

additive inverse $\forall a \in F, \exists a'$ s.t. $a + a' = 0$

additive ident. $\exists a \in F \ \forall a' \in F \ a' + a = a'$

These $|V|-1$ vectors defined by edges in T. These $|V|-1$ vectors are Lin. Ind. So, the dim. of this space is at least $|V|-1$.

$$\mathrm{cyc}(G) \leq |E| - (|V|-1)$$

$$= |E| - |V| + 1$$

$$\Downarrow$$

$$\mathrm{cyc}(G) = |E| - |V| + 1.$$

$\square$

Denote by $\delta_G(v)$ the decrease in the cyclomatic number of G on removing v. Since the removal of a feedback vertex $F = \{v_1, \ldots v_f\}$ brings $\mathrm{cyc}(G)$ down to zero.

$$\mathrm{cyc}(G) = \sum_{i=1}^{f} \delta_{G_{i-1}}(v)$$

where: $G_0 = G$

for $i > 0$: $G_i = G - \{v_1, v_2, \ldots, v_i\}$.

$\rightarrow \quad cyc(G) \leq \sum_{v \in F} \delta_G(v)$ by lemma below: $(\star)$

<u>Lemma 6.4</u>: If H is a subgraph of G, then
$$\delta_H(v) \leq \delta_G(v).$$

Let's say that the weight function is <u>cyclomatic</u> if there is a constant $c > 0$ s.t. the wt. of each vertex is $c \cdot \delta_G(v)$.

by $(\star)$

$$c \cdot cyc(G) \leq c \cdot \sum_{v \in F} \delta_G(v) = w(F) = OPT$$

we'll show that for any cyclomatic weight function, a minimal feedback vertex set has weight within twice the optimal.

Let $deg_G(v)$ denote degree of v in G.

Let $\text{comps}(G-v)$ = # conn. components in $G - \{v\}$.

<u>Claim:</u> For a connected graph G:

$$\delta_G(v) = \deg_G(v) - \text{comps}(G-v)$$

<u>Thm: 6.2:</u> $\text{cyc}(G) = |E| - |V| + \text{comps}(G)$

$$\text{cyc}(G) = |E| - |V| + 1$$

$$\text{cyc}(G-v) = \left(|E| - \deg_G(v)\right) - \left(|V| - 1\right) + \text{comps}(G-v)$$

$$\text{cyc}(G) - \text{cyc}(G-v) =$$

$$\cancel{|E|} - \cancel{|V|} + \cancel{1} - \cancel{|E|} + \deg_G(v) + \cancel{|V|} - \cancel{1} - \text{comps}(G-v)$$

$$= \deg_G(v) - \text{comps}(G-v)$$

<u>Lemma:</u> Let H be a subgraph of G [not necessarily vertex induced].

Then, $\delta_H(v) \leq \delta_G(v)$.

<u>Proof:</u> We only prove it for the connected components of G & H that contain v.

→ we assume G & H are connected.

To show:

$$\deg_H(v) - \text{comps}(H-v) \leq \deg_G(v) - \text{comps}(G-v)$$

Let $C_1, \ldots, C_k$ be components left over by removing v in H.

1) Edges of $G-H$ that are NOT incident at v.

$$\text{comps}(H-v) \geq \text{comps}(G-v)$$

2) Edges of $G-H$ that ARE incident at v.

These edges may add 1 to # comps, but its balanced out by its contribution to degree.

$$\deg_H(v) - \text{comps}(H-v) \leq \deg_G(v) - \text{comps}(G-v)$$

$$\Downarrow$$

$$\delta_H(v) \leq \delta_G(v) \quad \text{for } \cancel{\text{some}}\ \text{any} \text{ subgraph } H$$

$\square$

<u>Lemma:</u> If F is a minimal feedback vertex set of G, then

$$\sum_{v \in F} \delta_G(v) \leq 2 \cdot cyc(G)$$

<u>Proof:</u>

- Prove it for a connected graph G.

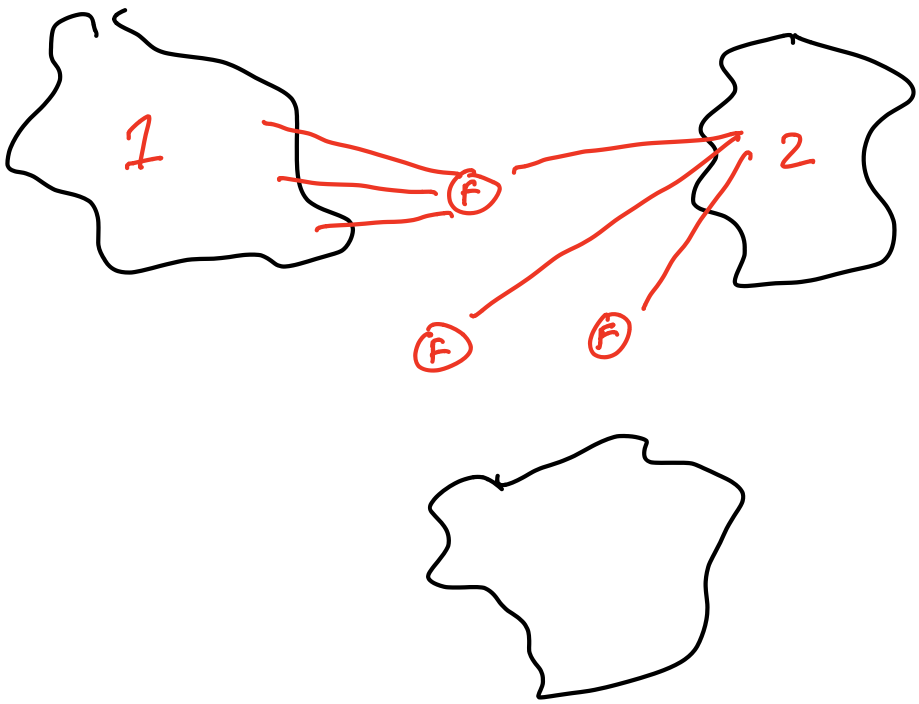
Let $F = \{v_1, \ldots, v_f\}$ and let k be the number of connected components after deleting F from G.

Partition these components into 2 types.

1) has edges incident to only 1 vertex in F

2) has edges incident to 2 or more vertices

in F



Say there are t components of type 1 and

$k-t$ components of type 2.

we will show:

$$\sum_{i=1}^{f} \delta_G(v_i) = \sum_{i=1}^{f} \left(\deg_G(v_i) - \mathrm{comps}(G - v_i) \right)$$

$$\leq 2\left(|E| - |U|\right) \leq 2\ \mathrm{cyc}(G)$$

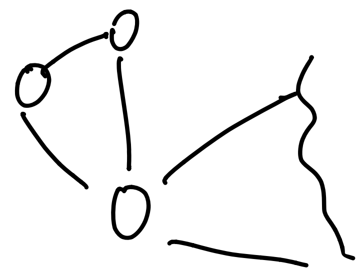
$Cyc(G) = |E| - |V| + comps(G)$

clearly: $\sum_{i=1}^{f} comps(G - v_i) = f + t$

↑ # components of type 1

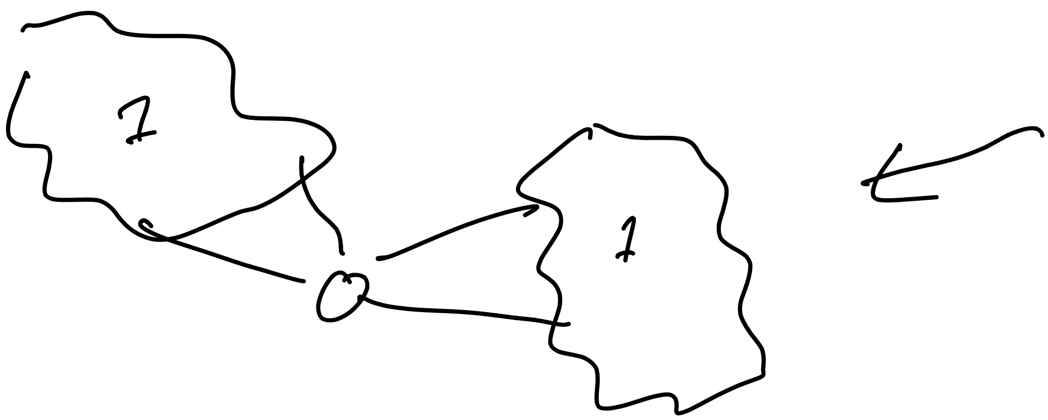
Induction: on $f = |F|$

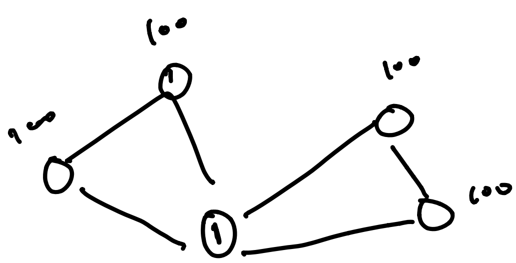
Base: $f = 1$



$comps(G - v) = t + 1$

2 2





Left to prove:

$$\sum_{i=1}^{f} \deg_G(v_i) \leq 2(|E|-|V|)+f+t$$

$$\sum_{i=1}^{f} \delta_G(v_i) = \sum_{i=1}^{f} \left(\deg_G(v_i) - \underbrace{\text{comps}(G-v_i)}_{=(f+t)}\right)$$

$$\leq 2(|E|-|V|) \leq 2\,\text{cyc}(G)$$

Since F is a FVS, these k components are all trees.

So, # edges in these k components.

$$(|V|-f)-k$$

Lower bound on # edges in the cut $(F, V-F)$

Since F is minimal, each $v_i \in F$ must be in some cycle that contains no other vertex from F.

So, each v_i must have at least two edges incident at one of these components.

For each v_i, arbitrarily remove one of these edges.

each of the t components still have at (type 1) least 1 edge in the cut $(F, V-F)$. Each of the

$k-t$ components still have at least 2 edges in the cut. (type 2)

$\#$ edges in cut $(F, V-F)$ is at least

$$f + t + 2(k-t) = f + 2k - t$$

$$\sum_{i=1}^{f} \deg_G(v_i) \leq 2|E| \underbrace{- 2(|V| - f - k) - (f + 2k - t)}$$

Goal: $\sum_{i=1}^{f} \deg_G(v_i) \leq 2|E| - 2|V| + f + t$

$$2|E| - 2|V| + 2f \cancel{+ 2k} - f \cancel{- 2k} + t$$

$$2\left(|E| - |V|\right) + f + t$$

$\square$

Corollary: $w : V \to \mathbb{R}_{\geq 0}$ cyclomatic wt. function.

F is a minimal FVS. Then,

$$w(F) \leq 2 \cdot OPT$$

$$w(v) = c \cdot \delta_G(v)$$

dec. in cyclomatic # by removing v from G

Given Graph $G = (V, E)$ and a wt. function w,

let

$$c = \min_{u \in V} \left\{ \frac{w(v)}{\underline{\delta_G(v)}} \right\}$$

$$t(v) = c \cdot \underline{\delta_G}(v)$$

largest cyclomatic weight function in w

$w'(v) = w(v) - t(v)$ residual weight function

$= w(v) - c \cdot \delta_G(v) = w(v) - \frac{w(v)}{\delta_G(v)} \cdot \delta_G(v)$

Let V' be the set of vertices with a positive residual wt. function. value.

$V' \subset V$.

Let G' be the subgraph of G induced on V'.

Using operation above, decompose G into nested subgraphs,

$t_i(v)$ cyc. function on G_i

w'

acyclic

Let these graphs be $G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_k$

G_i is the induced subgraph on vertex set V_i

where $V = V_0 \supset V_1 \supset V_2 \supset \ldots \supset V_k$.

Let t_i for $i = 0 \ldots k-1$ be the cyclomatic weight function for graph G_i.

$w_0 = w$ the residual weight function for G_0

t_0 largest cyclomatic wt. function for w

$w_1 = w_0 - t_0$

End, w_k residual wt. function for acyclic G_k.

let $t_k = w_k$.

the weight of vertex v is decomposed into the weights $t_0, t_1, t_2 \ldots$

$$\sum_{i=0}^{k} t_i(v) = w(v)$$

Lemma 6.7: Let H be a subgraph of $G=(V,E)$ on vertex set $V' \subset V$. Let F be a minimal FVS on H. Let $F' \subseteq V - V'$ be a

~~the~~ minimal set s.t. $F \cup F'$ is a FVS for G, then $F \cup F'$ is a minimal FVS for G.

Proof: Let $v \in F$ be some vertex. Since F is minimal, there must be some cycle C that uses v but no other vertex from F.

we know, $F' \subseteq V - V'$ so $F' \cap V' = \emptyset$.

So, C uses only the vertex v $F \cup F'$.

So, $F \cup F'$ is minimal.

$\square$

LAYERING

Algorithm for FVS:

1. Decomposition phase

$H \leftarrow G$, $w' \leftarrow w$, $i \leftarrow 0$

while H is not acyclic

$$c \leftarrow \min_{v \in V} \left\{ \frac{w'(v)}{\delta_H(v)} \right\}$$

$G_i \leftarrow H$, $t_i \leftarrow c \cdot \delta_{G_i}$, $w' \leftarrow w' - t_i$

$H \leftarrow$ subgraph of G_i induced by vertices v s.t. $w'(v) > 0$

$i \leftarrow i+1$

$k \leftarrow i$, $G_k \leftarrow H$

2. $F_k \leftarrow \emptyset$

F_i is FVS for G_i

For $i = k, \ldots, 1$: extend F_i to a FVS for F_{i-1} by adding a minimal set of vertices from $V_{i-1} - V_i$.

Output F_0.

F_0 is FVS for $G_0 = G$

<u>Thm:</u> factor 2 approx.

<u>Proof:</u> Let F^* be an optimal FVS for G. Since G_i is an induced subgraph of G, $F^* \cap V_i$ must be a FVS for G_i [not necessarily the best for G_i]. Since, the weights of vertices have been decomposed:

$$OPT = w(F^*) = \sum_{i=0}^{k} t_i (F^* \cap V_i) \geq \sum_{i=0}^{k} OPT_i$$

when OPT_i is optimal FVS for G_i.

our alg $\to F_0$.

$$w(F_0) = \sum_{i=0}^{k} t_i (F_0 \cap V_i) = \underbrace{\sum_{i=0}^{k} t_i (F_i)}$$

We know F_i is a minimal FVS for G_i.

We know $0 \le i \le k-1$, t_i is a cycl. wb. function.

by lemma 6.5 $t_i(F_i) \le 2 \cdot OPT_i$

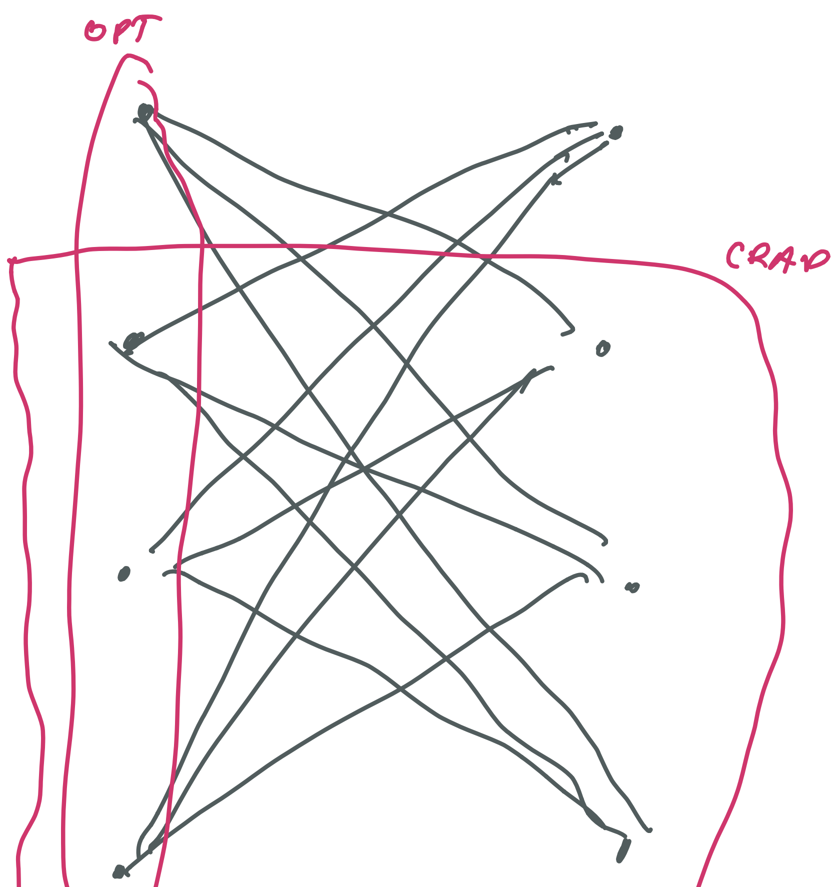
$F_k = \emptyset$.

$$w(F_0) = \sum_{i=0}^{k-1} t_i(F_i) \le \sum_{i=0}^{k-1} 2 \cdot OPT_i = 2 \cdot \sum_{i=0}^{k-1} OPT_i$$

$$\le 2 \cdot OPT$$

$\square$

Tight example:



Thx.